State Feedback Control in Macroeconomic Policy

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Abstract

Many of the currently used monetary policy rules are of the form of proportional controllers. We extend existing models to consider a different type of control–steady state feedback. We propose a general form for the state feedback controllers with parameters which can be adjusted to reflect the macroeconomy in question. Our controller offers an improvement over existing strategies in time to reach equilibrium and smoothness. However, it is limited by the need apply high interest rates and access accurate information about high-order derivatives of inflation.

1 Introduction

Control theoretic approaches to macroeconomic policy have been examined since the 1950's [1, 2] with the use of feedback as a stabilization mechanism introduced in 1954 [3]. The 1970's saw criticism of the use of control theory in macroeconomic policy design [4]. Kyland and Prescott concluded that "there is no way control theory can be made applicable to economic planning when expectations are rational" [5]. They argued that economic models did not exhibit time-invariant behavior, and were therefore poor candidates for optimal control. Those criticisms were later revised [6, 7], with rational expectations becoming a consideration rather than an insurmountable obstacle [8–17].

In their 2014 paper, Hawkins et al. noted the similarity between PID controllers and the macroeconomic regulation policies in use by central banks [18]. This paper builds on their work by considering a different type of control: full state feedback. Full state feedback offers a few key advantages over other control strategies because it allows the user to direct the behavior of derivative parameters and provides the ability to directly set various system characteristics, such as pole location [19]. However, state feedback control is greatly limited by the need for accurate information about parameters which may be difficult to estimate with high precision.

The remainder of this paper is organized as follows. In Section 2, we present the current approach to macroeconomic policy: a convex optimization problem which minimizes the central banks loss function subject to the Phillips curve. The solution to this formulation is the Monetary Policy Rule. We then estimate the Monetary Rule's performance in simulation. In Section 3, we proceed to an alternative approach for optimizing macroeconomic performance. We formulate the macroeconomy as a continuous-time system and derive a full state feedback controller which minimizes a cost function. We will then compare the performance of this controller with that of the traditional approach and evaluate its advantages and limitations. Section 4 concludes and offers some directions for future work. Derivations can be found in Section 5, the Appendix.

2 The Canonical Approach

2.1 The Three Equation Model

New Keynesian macroeconomic dynamics is based on three key relationships: the Phillips Curve (PC), which captures the relationship between inflation and output, the Investment-Savings (IS) curve, which relates output and interest rate, and the interest-based monetary policy rule. The Monetary Rule is derived from a convex optimization problem based on the minimizing a central bank loss function subject to the Phillip's Curve:

$$\pi_1 = \pi_0 + \alpha(y_1 - y_e) \tag{1}$$

where the present time inflation rate, π_0 is related to the following period inflation rate π_1 by the difference between next period output y_1 and the target output y_e and α, which captures the responsiveness of inflation to changes in output. The IS curve is

$$y_1 = A_0 + ar_0 \tag{2}$$

where A_0 is autonomous expenditure at current time, r_0 is the current interest rate, and a captures the sensitivity of investment to the real interest rate. The loss function

$$L = (y_1 - y_e)^2 + \beta(\pi_1 - \pi^T)^2 \tag{3}$$

illustrates the loss experienced by the central bank due to deviations from equilibrium output y_e and target inflation π^T at the next period.

2.2 Gapped Notation

Different economies have different targets for inflation. This makes the model presented above cumbersome when considering different nations or even particular points in time. Instead, we will use a model with equivalent dynamics but which considers the evolution of the gaps between inflation, rate, and output and their targets rather than the three parameters themselves.

Subtracting the desired value of inflation, π^T, from both sides of equation (1), we have

$$\pi_1 - \pi^T = \pi_0 - \pi^T + \alpha(y_1 - y_e) \tag{4}$$

The terms $\pi_1 - \pi^T$ and $\pi_0 - \pi^T$ capture the deviation of inflation from its target in the current and following period, respectively, so we will replace these with the terms $\pi_g[n]$, $\pi_g[n-1]$, the gaps between inflation and target inflation at time n and time $n-1$. Similarly, $y_1 - y_e$, the deviation from output at time 1 will be replaced with $y_g[n]$. Performing similar substitutions on the IS curve and loss function gives us the following set of equations[1]:

$$\pi_g[n] = \pi_g[n-1] + \alpha y_g[n] \tag{5}$$

$$y_g[n] = -a r_g[n] \tag{6}$$

$$L[n] = y_g[n]^2 + \beta \pi_g[n]^2 \tag{7}$$

[1]Henceforth, we will follow the convention of using square brackets and the variable n to denote discrete time and parenthesis and the variable t for continuous time.

2.3 Ratio-Based Weighting

In addition to different targets, different economies are guided by different emphasis on output and inflation. To account for economies with different types of emphasis, we will further modify the model presented in section 2.1 to consider emphasis on one variable or another. We will then provide a derivation of the monetary policy rule with this model. Finally, we perform simulations with various weights on the two parameters to gain some intuition into the impact of emphasizing one consideration or another and form a basis of comparison with our controller.

We start by rewriting the central bank loss function with new weights, β_1 and $1-\beta_1$, where β_1 varies from 0 to 1 and expresses the proportion of emphasis on meeting inflation targets as opposed to output targets. For example, a β_1 value of 1 would indicate a central bank which was solely concerned with achieving target inflation and ignores deviation from output in their evaluation of the quality of economic performance.

$$L[n] = (1-\beta_1)y_g[n]^2 + \beta_1\pi_g[n]^2 \tag{8}$$

We minimize the loss function subject to the Phillip's Curve

$$\pi_g[n] = \pi_g[n-1] + \alpha y_g[n] \tag{9}$$

We form the Lagrangian for the problem as

$$\mathcal{L} = L[n] - \lambda\left[-\pi_g\left[n\right] + \alpha y_g\left[n\right] + \pi_g\left[n-1\right]\right] \tag{10}$$

Differentiating with respect to $y_g[n]$ and $\pi_g[n]$ we have

$$\frac{\partial \mathcal{L}}{\partial y_g} = 2(1-\beta)y_g[n] - \alpha\lambda = 0 \tag{11}$$

or

$$2(1-\beta)y_g[n] = \alpha\lambda \tag{12}$$

and

$$\frac{\partial \mathcal{L}}{\partial \pi_g} = 2\beta\pi_g[n] + \lambda = 0 \tag{13}$$

or

$$2\beta\pi_g[n] = -\lambda \tag{14}$$

Clearing λ we have

$$\frac{(1-\beta)y_g[n]}{\alpha} = -\beta\pi_g[n] \tag{15}$$

or

$$y_g[n] = \frac{-\beta\alpha\pi_g[n]}{(1-\beta)} \tag{16}$$

We use this relationship to rewrite the Phillip's Curve so that:

$$\pi_g[n+1] = \pi_g[n] + \alpha y_g[n+1]$$

becomes

$$\pi_g[n+1] = \pi_g[n] + \alpha\frac{-\beta\alpha\pi_g[n+1]}{1-\beta}$$

so

$$\pi_g[n+1]\Big(1 + \frac{\alpha^2\beta}{1-\beta}\Big) = \pi_g[n]$$

or

$$\pi_g[n+1] = \frac{\pi_g[n](1-\beta)}{1-\beta+\alpha^2\beta}$$

We now want to rewrite the rate gap

$$r_g[n] = \frac{1}{a}\frac{\alpha\beta\pi_g[n+1]}{1-\beta}$$

in terms of the current inflation gap. Using the relationship we found above, this is

$$r_g[n] = \frac{\alpha\beta\pi_g[n]}{a(1-\beta+\alpha^2\beta)}$$

With these new weights, we can explore the effects of targeting inflation and output equally or differently. Setting $\beta = .75$, for example, represents inflation targeting whereas $\beta = .25$ represents output targeting. The response of the inflation rate, interest rate, and output deviation in percent to an inflation shock given varying targeting goals is shown in Figure 1.

Comparing the lowest curves in the two bottom panels of Figure 1 we see how an inflation targeting bank will implement rate policy which causes a quicker convergence but results in greater undershoot in output. Conversely, an output targeting bank will experience slower convergence, but much less deviation in output. The Federal Reserve—the central bank of the US—obeys a "dual mandate", that is, in contrast to single-mandate inflation targeting banks, the central bank must target inflation and output equally [20].

In equal targeting, as shown by the top panel of Figure 1, convergence takes approximately two years and the interest rate is at most about 0.5 above target (for the U.S., this represents an interest rate of 2.5%, as the target rate is 2% [20]). These values will be used as a basis of comparison for the controller derived in the following section.

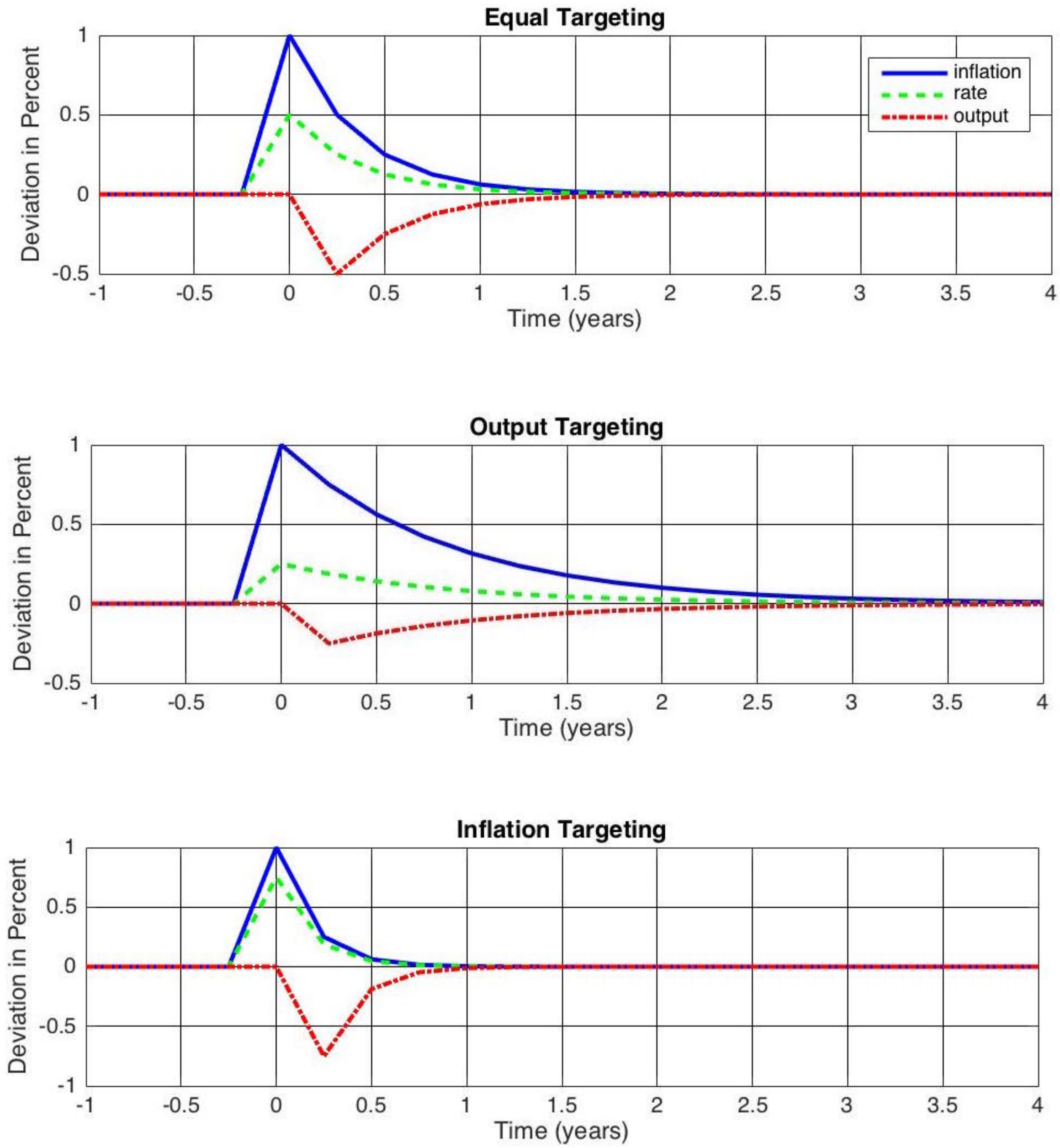


Figure 1: Responses to inflation given equal, output, and inflation targeting

3 State Feedback Control

3.1 Transfer Function Derivation

In order to design our controller, we will consider our system as represented in the time domain by the IS and Phillips Curves. From this we will derive the frequency domain transfer function, and then move back to the time domain by considering the system's state space representation.

Although it would be possible to bypass the frequency domain and move directly from the time domain equations governing the system to its state space representation, we consider the transfer function a useful formulation for several reasons. A transfer function is a ratio of a system's output to its input as polynomials in frequency. Whereas in the time domain, the system is represented by a differential equation, the transfer function represents it as an algebraic equation [19]. This is particularly helpful when considering multiple systems in combination. In the time domain, the cascade of multiple systems requires convolution, but in frequency the resultant system is found through multiplication of the algebraic transfer functions [21]. This makes analysis significantly easier and more intuitive, and can provide insight into the behavior of the system which, while equivalently presented in the time domain, is less apparent.

We begin with the continuous time representation of the PC and IS [22] [23]:

$$\pi_g(t) = \int_{-\infty}^{t} \chi_y(t-\tau) y_g(\tau) d\tau \tag{17}$$

$$y_g(t) = \int_{-\infty}^{t} \chi_r(t-\tau) r_g(\tau) d\tau \tag{18}$$

where $\pi_g(t), y_g(t)$ and $r_g(t)$ are the continuous time gaps between the actual and target values for inflation, output, and interest rate, respectively. Since these functions are zero valued before time 0 they are equivalent to convolution. The central bank's loss function can be written

$$L(t) = \beta \int_0^t \pi_g(\tau)^2 d\tau + \int_0^t y_g(\tau)^2 d\tau \tag{19}$$

While we will not make use of equation 19 in this paper, we provide this formulation of the loss function because it may be interesting to consider the problem as a convex optimization analogous to the discrete time optimization in Section 2.

We now consider these relationships in the frequency domain. Since equations (17) and (18) depict a time-domain convolution, we have a frequency domain multiplication of these quantities [21]. In the frequency domain, the PC and IS are:

$$G_\pi(s) = \chi_y(s) G_y(s) \tag{20}$$

$$G_y(s) = \chi_r(s) G_r(s) \tag{21}$$

Combining equations (20) and (21) we have a relationship between the interest rate, our input, and inflation, our output [2].

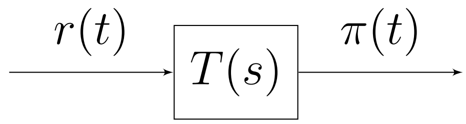
$$G_\pi(s) = \chi_y(s) \chi_r(s) G_r(s) \tag{22}$$

[2] In this paper, we use the word output in two distinct ways. In Section 2, output is used to mean economic output, usually measured in GDP. In Section 3, the word "output" refers to the variable returned by the system, which in our case is inflation

So the system transfer function is

$$T(s) = \chi_y(s)\chi_r(s) \tag{23}$$

where the system is shown below.



Let us now write $T(s)$ as a ratio of two polynomials in s. The output response function $\chi_y(t)$ is [22]:

$$\chi_y(t) = \frac{1}{\omega_1 m} e^{-\gamma t/2} \sinh(\omega_1 t) \tag{24}$$

which, in the frequency domain, is [21]:

$$\chi_y(s) = \frac{1}{\left(\left(s + \frac{\gamma}{2}\right)^2 - \omega_1^2\right) m} \tag{25}$$

The rate response function $\chi_r(s)$ is the solution to the differential equation [23] :

$$\tau_r \frac{dy(t)}{dt} + y(t) = \tau_r J_u \frac{dr(t)}{dt} + J_r r(t) \tag{26}$$

Performing a Laplace transform on equation (26), we obtain

$$\tau_r s Y(s) + Y(s) = \tau_r J_u s R(s) + J_r R(s) \tag{27}$$

which gives us

$$\chi_r(s) = \frac{\tau_r J_u s + J_r}{s\tau_r + 1} \tag{28}$$

So our transfer function is

$$T(s) = \frac{\tau_r J_u s + J_r}{(s\tau_r + 1)\left(\left(s + \frac{\gamma}{2}\right)^2 - \omega_1^2\right) m} \tag{29}$$

However, because the instantaneous part of rate response is not usually observable [23], $J_u = 0$. This gives us

$$T(s) = \frac{J_r}{(s\tau_r + 1)\left(\left(s + \frac{\gamma}{2}\right)^2 - \omega_1^2\right) m} \tag{30}$$

Or, in standard form

$$T(s) = \frac{J_r}{m\left(\tau_r s^3 + (\tau_r\gamma + 1)\, s^2 + \left(\frac{\gamma^2\tau_r}{4} - \omega_1^2\tau_r + \gamma\right) s + \frac{\gamma^2}{4} - \omega_1^2\right)} \tag{31}$$

3.2 State Space Representation

In a dynamical system, the states, collected in the state space vector x, are a set of variables that characterize the evolution of the system. In our system, for example, the states x_1, x_2, and x_3 are inflation and its first and second derivative, because the behavior of the system and its development over time can be completely predicted from information about the inflation rates and its derivatives [24].

For linear and time invariant systems, behavior can be characterized in state space using the dynamics matrix A, the control matrix B, the sensor matrix C, and the direct term D, where the total system has the form $\dot{x} = Ax + Bu$, $y = Cx + Du$. In our case, the control signal u is the interest rate set by a nation's central bank, and the output of the system y

is inflation [3]. Since the output is identical to the first entry of the state vector, the matrix D will be 0 in our case. The other matrices are presented below. Their derivation from the transfer function can be found in the Section 5.1.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{\tau_r\gamma+1}{\tau_r} & -\frac{\frac{\gamma^2\tau_r}{4}-\omega_1^2\tau_r+\gamma}{\tau_r} & -\frac{\frac{\gamma^2}{4}-\omega_1^2}{\tau_r} \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 0 \\ \frac{J_r}{\tau_r m} \end{bmatrix} \tag{32}$$

and

$$C = [1 \quad 0 \quad 0] \tag{33}$$

3.3 Controllability

Controllability is a system property wherein externally applied input (in our case, the control signal u) is able to alter the initial state of the system in finite time [19]. State feedback can only be applied to systems which are controllable. This section demonstrates the controllability of the system under consideration, thus showing that state feedback is in fact a valid approach to achieve desired behavior.

A system is considered controllable if its controllability matrix is full rank [25]. The controllability matrix of our system is given by

$$C_m = \begin{bmatrix} B & AB & A^2B \end{bmatrix} \tag{34}$$

[3] As explained in footnote (2), "output" here refers to the result of the system, not the economic quantity.

Which is

$$C_m = \begin{bmatrix} 0 & 0 & \frac{J_r}{\tau_r m} \\ 0 & \frac{J_r}{\tau_r m} & \frac{(\omega_1^2 - \frac{\gamma^2}{4})J_r}{\tau_r^2 m} \\ \frac{J_r}{\tau_r m} & \frac{(\omega_1^2 - \frac{\gamma^2}{4})J_r}{\tau_r^2 m} & \frac{J_r(-4\gamma^2\tau_r^2 + 16\omega^2\tau_r - 16\gamma\tau_r + \gamma^4 - 8\gamma^2\omega_1^2 + 16\omega_1^4)}{16\tau_r^3 m} \end{bmatrix} \tag{35}$$

Since we have a square controllability matrix, we know that the matrix is full rank if the determinant is non-zero [26]. The determinant of our matrix is

$$|C_m| = -\frac{J_r^3}{\tau_r^3 m^3} \tag{36}$$

If this seems remarkable, note that interchanging the first and third rows of the controllability matrix results in an upper triangular matrix, whose determinant is simply the diagonal entries multiplied together [26]. Thus, the system is controllable for for any finite values of τ_r and m and any non-zero value of J_r. The usual range of these parameters satisfies the constraint [22, 23], so macroeconomic systems are, in general, controllable.

3.4 State Feedback Control

Having established the controllability of our system, we will now introduce state feedback control of the form $u = r - [k1 \quad k2 \quad k3]\,x$ to attain desired dynamic behavior. There are two main approaches to gain selection. The first option allows us to place the poles directly, by selecting desired overshoot and settling time behavior and analytically choosing our poles to meet this criteria [19]. An alternative is to use a Linear-Quadratic-Regulator (LQR). The first use of Linear Quadratic techniques in a maroeconomic setting was presented by Pindyck in 1972 [27]. In LQR control, gains are chosen to minimize a quadratic cost function. The costs are captured by two positive semi definite matrices Q_x and Q_u [24]. While there are various forms for the matrices, we chose to use diagonal matrices whose entries therefore capture how much each state and input contributes to the overall cost.

The simulations were performed in MATLAB using the state space model object (with the command `ss(A, B, C, D)`) created from the matrices derived in section 3.3. The Q matrices were varied by increasing each parameter by a factor of 10 while holding the others constant. The optimal gains for each choice of Q_u, Q_r were calculated by minimizing the quadratic cost function using the command `lqr`. Because our goal was to achieve zero deviation from target rates of inflation, the reference input signal $r(t)$ was zero. A 1% inflation shock was represented by a vector of initial conditions $[1 \quad 0 \quad 0]$, and simulated using the command `initial`. This command returns vectors describing the evolution of output and the states over time, giving us direct access to the inflation response. Conversely, interest rate is not directly returned, but was reconstructed by multiplying the calculated optimal gain value k_i by the corresponding state variable x_i and summing the results. The table below lists the parameter values used for our simulation [22, 23]:

Parameter	**Value**
γ	4.37
ω_1	$.93 \times 10^{-2}$
τ_r	11.6279
m	640.36
J_r	-1.9186

The images in the left column of Figure 2 illustrate the response of the system with varying cost matrices. We found that overall there is a trade-off between interest rate and convergence speed. The result of emphasizing inflation cost is shown in the panel in the second row. This entry has by far the highest peak interest rate: approximately 2.75 % as opposed to maxima of 0.65 % in equal emphasis weighting (top panel) and inflation derivative weighting (third row panel) and 0.1 % in interest rate emphasis weighting (bottom panel). While the panels on equal cost emphasis and change in inflation cost emphasis have similar peak values of rate, their minimal rate values and their convergence speeds differ. The interest rate in the top panel falls to slightly below 0 %, while the interest rate in the third row panel reaches below -0.2 %. Because the controller in the third row attempts to minimize the derivative of inflation, it is trying to achieve gradual change. Therefore, it more aggressively reduces the interest rate after the initially high value in order to slow the descent of inflation. As expected, the derivative-emphasizing controller takes longer to converge: approximately 15 years to the equal weighting controller's 8 years. Like the derivative-emphasis controller, the rate emphasizing one takes approximately 15 years to converge to target. However, it keeps the interest rate much lower, and lets it oscillate so that inflation descends by passing through periods of fast and slow decrease. Although the optimal choice of cost values is outside the scope of this paper, the panels in Figure 2 provide some intuition into the consequences of different weight choices.

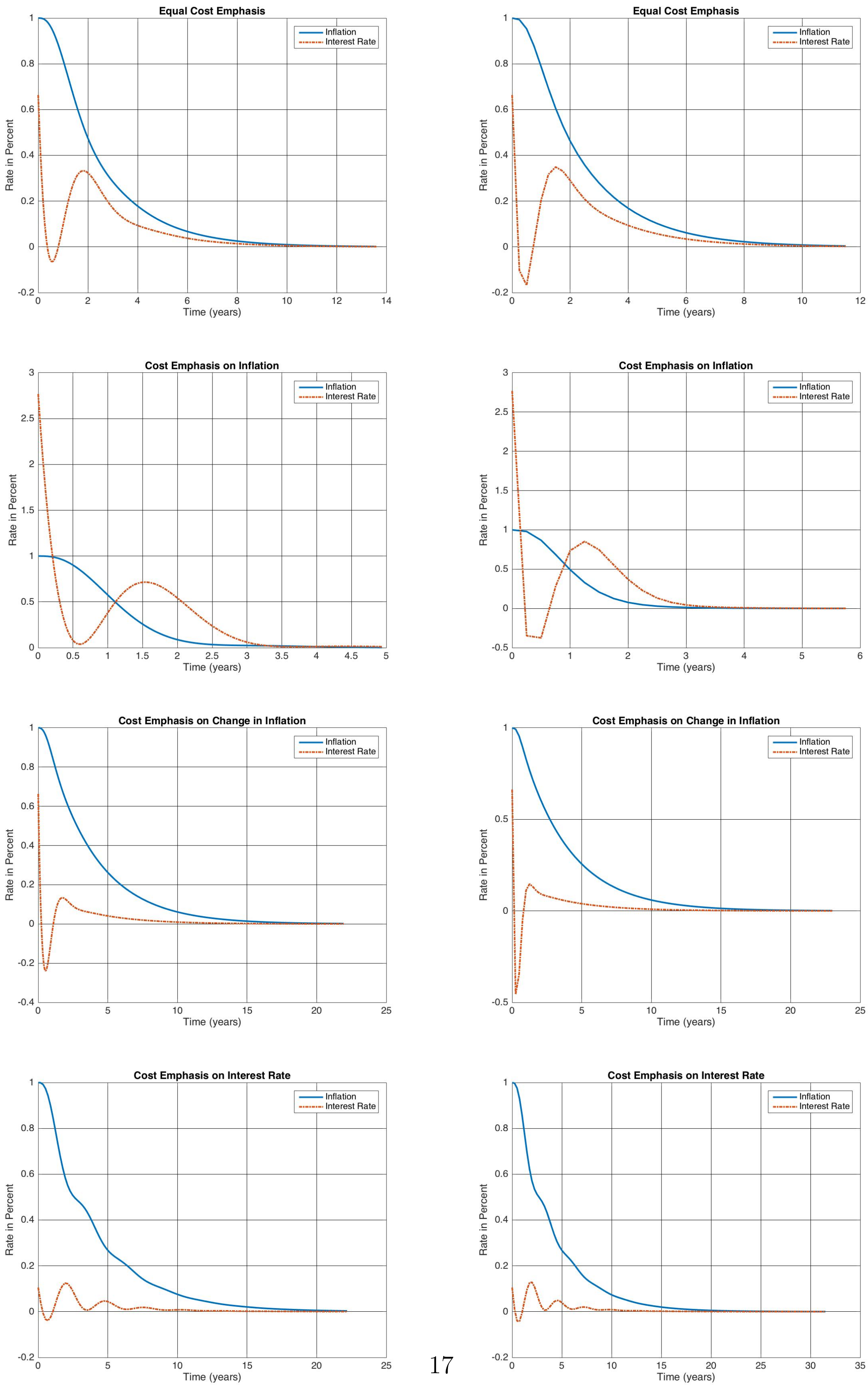


Figure 2: Continuous- (left) and discrete- (right) time responses with varying cost matrices

3.5 Discretization

In this section, we will consider the discretized version of our system in order to form a better basis of comparison against the Monetary Rule considered in Section 2. Because the federal reserve makes control decisions on a quarterly basis, we consider the economy as a discretized version of the model presented in section 3.5 with a sampling time $T_s = .25$ years. This introduces discretization error [28]. The discrete-time response to an inflation shock given various cost functions is shown in the right hand column of Figure 2.

The discrete time system performance is similar to the continuous time performance. The primary differences are that discrete time rate tends to experience greater undershoot and slightly faster convergence. The inflation cost emphasizing controller's results most closely resemble those of the Rule presented in Section 2. Our controller offers a decreased time to convergence by several periods. It also provides a smoother transition to the equilibrium rate. This is a reasonable improvement, considering that our controller takes into account derivative parameters which are not considered by the Monetary Rule. As in continuous-time, the major limitation of our controller's performance is that it requires higher interest rates to achieve targets.

4 Conclusion

This work considers a macroeconomic system as represented by the discrete and continuous time Phillip's Curve and Investment Savings Curve. A nation's central bank choses an interest rate to obtain optimal behavior. We consider the choice of interest rate from two different perspectives: as a convex optimization minimizing the Central Bank Loss function or alternatively as an LQR controller choosing gain to minimize a cost function representing deviation from target rate, inflation, and the first two derivatives of inflation.

Our LQR controller offers a few advantages over the Monetary Rule. It shows a faster convergence to equilibrium, whose speed can be increased by increasing the weight on the cost matrices presented in Section 3.5. The controller also provides a smoother convergence, because its cost function considers the derivative of inflation, while the Monetary Rule does not. Our controller also offers a novel way to achieve lowered inflation while keeping interest rates very low: an oscillatory interest rate. Although this strategy also leads to oscillation in the derivative of inflation, that is, the descent is "bumpy", it converges to the target in approximately the same amount of time as a strategy using much higher interest rates.

Our controller has several key limitations. In general, it requires higher interest rates than does the Monetary Rule to achieve the same performance. In particular, convergence below a certain threshold (approximately half a year) requires that interest rates oscillate with high magnitude years after inflation has reached target. Another major limitation is that it is impossible to directly access information about the derivatives of inflation. In practice, a numerical approximation would be used to estimate derivatives, and it would not have high accuracy. We do not have an estimation for the exact impact on controller performance, however, the problem is likely to be severe, especially in the rate emphasizing scheme where derivatives change non-monotonically. This problem can be addressed by incorporating an observer into the system, which should greatly increase the accuracy of state variable estimates.

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5 Appendix

5.1 Transfer Function to State Space

In this section, we will demonstrate the conversion from transfer function representation to state space representation. The transfer function

$$T(s) = \frac{J_r}{m\left(\tau_r s^3 + (\tau_r\gamma + 1)\, s^2 + \left(\frac{\gamma^2\tau_r}{4} - \omega_1^2\tau_r + \gamma\right) s + \frac{\gamma^2}{4} - \omega_1^2\right)} \tag{37}$$

corresponds to the time-domain differential equation

$$m\tau_r \frac{d^3}{dt^2}\pi(t) + (m\tau_r\gamma + 1)\frac{d^2}{dt^2}\pi(t) + m\left(\frac{\gamma^2\tau_r}{4} - \omega_1^2\tau_r + \gamma\right)\frac{d}{dt}\pi(t) + \left(\frac{\gamma^2}{4} - \omega_1^2\right)\pi(t) = J_r r(t) \tag{38}$$

Let us chose the state x_1 to be the output, and the subsequent states to be derivatives of each other. That is

$$x_1 = \pi \tag{39}$$

$$x_2 = \frac{d\pi}{dt} \tag{40}$$

$$x_3 = \frac{d^2\pi}{dt^2} \tag{41}$$

where taking the derivative of the above yields

$$\dot{x_1} = \frac{d\pi}{dt} \tag{42}$$

$$\dot{x_2} = \frac{d^2\pi}{dt^2} \tag{43}$$

$$\dot{x_3} = \frac{d^3\pi}{dt^3} \tag{44}$$

From this, we can see that

$$\dot{x_3} = -\frac{\tau_r\gamma + 1}{\tau_r}x_3 - \frac{\frac{\gamma^2\tau_r}{4} - \omega_1^2\tau_r + \gamma}{\tau_r}x_2 - \frac{\frac{\gamma^2}{4} - \omega_1^2}{\tau_r}x_1 + \frac{J_r}{\tau_r m}u \tag{45}$$

from which we obtain the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{\tau_r\gamma+1}{\tau_r} & -\frac{\frac{\gamma^2\tau_r}{4}-\omega_1^2\tau_r+\gamma}{\tau_r} & -\frac{\frac{\gamma^2}{4}-\omega_1^2}{\tau_r} \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 0 \\ \frac{J_r}{\tau_r m} \end{bmatrix} \tag{46}$$

and

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{47}$$

as desired.