Theory Field Examination
Game Theory (209A)
Jan 2010

Good luck!!!

Question 1 (duopoly games with imperfect information)

Consider a duopoly game in which the inverse demand function is linear where it is positive

$$P(Q) = \begin{cases} \alpha - Q & \text{if} \quad Q \leq \alpha \\ 0 & \text{if} \quad Q > \alpha \end{cases}$$

where $Q = q_1 + q_2$ and q_i is firm i's output (both firms produce the same good). Firm i's profit is given by

$$\pi_i(q_1, q_2) = q_i P(Q) - C_i(q_i).$$

where $C_i(q_i)$ is the cost of producing q_i units. Assume that unit cost is constant $C_i(q_i) = c_i q_i$ for all q_i and that each firm has probability μ of having unit cost c_L and $(1 - \mu)$ of having cost $c_H > c_L$. Each firm knows only its unit cost. Each firm only knows its unit cost.

Solve for the Bayesian Nash equilibrium of the Cournot game, and for the sequential equilibrium of the Stackelberg's game (firm 1 moves at the start of the game).

Question 2 (trembling hand perfection)

Consider the following three-player game (van Damme, 1983) in which player 1 chooses rows, player 2 chooses columns, and player 3 chooses boxes

l

	L	R
T	$1,1,1$	$1,0,1$
B	$1,1,1$	$0,0,1$

r

	L	R
T	$1,1,0$	$0,0,0$
B	$0,1,0$	$1,0,0$

Show that the game has a Nash equilibrium in which no action is weakly dominated but it is *not* a trembling hand perfect equilibrium.

Best wishes for 2010!

Part I

a) Sami has two possible career moves (he must choose exactly one):

He could take a job as a forklift operator, and earn $\$\frac{3}{16}$.

Or he could become a spoonlift operator, and earn \$0 — no money.

(These wages come from a strange labour-market equilibrium in Sami's country. And if you don't know what these two jobs are, they are irrelevant to the problem.)

Sami's utility is given by $u = w + p^2$, where w is his income and p is his probabilistic beliefs (after retiring from his job) that he is a good spoonlift operator. (He either knows or doesn't care how good a forklift operator he is.) Sami (and all other characters in this problem) wishes to maximize the expected value of this utility.

Sami has (rational) priors $\overline{p} \in [0, 1]$ that he is good with spoonlift operating. If he takes the job as forklift operator, he never updates those beliefs. If he takes the spoonlift job, he will learn one way or another for sure whether he is good at it.

Write down (and circle) clearly Sami's expected utility for each of the two jobs.

b) For what values of $\overline{p} \in [0, 1]$ will Sami take the job as a spoonlift operator? Give a brief intuition for what you find. (And, as always, you can 'cheat' by double checking your math answers by using your intuition!)

c) Hamlet is also choosing between these same two careers. But he has a different utility function, and his labour market is different. He, like Sami, only cares about money and how good a spoonlift operator he is, but his utility is $u = w + \varphi p$, where φ might be positive, negative, or zero. (He might hate the idea of being a good spoonlift operator.) Like Sami, Hamlet learns whether he is good at spoonlifting if and only if he takes the spoonlifting job.

Hamlet's market is such that: if he takes the forklifting job, he earns w_f. If he takes the job spoonlifting, he earns w_s^g if he is good at spoonlifting, but $w_s^b < w_s^g$ if he is bad at it.

As a function of φ, w_f, w_s^b, and w_s^g, for what values of prior beliefs $\overline{p} \in [0, 1]$ will Hamlet take a job as a spoonlift operator? Give a succinct intuition for your answer and how it depends on the various variables. (And if you cannot solve the math, also give a succinct intuition for what you'd expect.)

d) Without doing math (I'm not giving you enough to solve it anyhow!), give a *brief* intuition for how your answer might change if Hamlet's performance as a spoonlifter provides only a noisy signal (to be both Hamlet and those paying him...) of whether he good at spoonlifting. Don't spend much time on this part.

e) Now consider Ophelia, who is also choosing which of these two careers to pursue. She has much the same concerns and Sami and Hamlet, but her utility is $u = w + z(p_1 - p_0)$, where $p_0 \equiv \overline{p}$ is her initial belief that she is good at spoonlifting, and p_1 are her final beliefs, after retiring, that she is good at spoonlifting, and $z(x) = \varphi x$ for $x \geq 0$ and $z(x) = -3\varphi x$ for $x \leq 0$.

She will never find out if she is good if she takes the forklifting job, but learn for sure whether or not she is good if she takes the spoonlifting job.

She gets paid w_f if she works forklifting, and w_s (irrespective of how good she is) if she spoonlifts. For what combinations of φ, w_f, w_s, and $p_0 \in [0, 1]$ will Ophelia take a job as a spoonlift operator? Give a succinct intuition for your answer and how it depends on the various variables.

Part II

f) Now consider "Intertemporal Ophelia": her utility in every period t is $u_t = w_t + z(p_t - p_{t-1})$, where p_k is her probabilistic beliefs at the end of period k about whether she is a good spoonlift operater, and as above $z(x) = \varphi x$ for $x \geq 0$ and $z(x) = -3\varphi x$ for $x \leq 0$. She is born with beliefs p_0, and will live forever beginning period 1. At the beginning of each period, IO (as her friends call her) chooses between working as forklift operator or as a spoonlifter. She can switch as often as she wants.

Unless and until she takes a job as spoonlifter, she does not update her beliefs about whether she is good at it. But once she takes the spoonlifting job, she learns that period for sure whether she is good, forever and ever knowing what she has learned, irrespective of whether she keeps spoonlifting in the future. *Each period*, she gets paid w_f if she works forklifting, and w_s (irrespective of how good she is) if she spoonlifts.

IO is, unfortunately, present-biased. Her goal each period t is to maximize $U^t \equiv u_t + \beta \sum_{\tau=t+1}^{\infty} \delta^{\tau-t} u_\tau$, where $\beta \leq 1$ and $\delta \leq 1$ are her discount factors. IO is however complete naive about her future self-control problems, believing $\widehat{\beta} = 1$, that in the future she will have full self control.

Characterize, as a function of β. φ, w_f, w_s, and p_0, what IO will do each period (including whether and when she will swictch jobs, etc., in the limit as $\delta \to 1$. Per always, don't worry about knife-edge cases of indifference. And characterize what happens fixing the parameters in the limit $\delta \to 1$; you don't need to say what will happen in the non-limiting case. Give an intuition for your answer. And, as a function of β, φ, w_f, w_s, and p_0, what is IO thinking and planning each period.

g) Now suppose that instead of being niave, IO is extremely sophisticated: she believes correctly that her future self-control problem will be $\widehat{\beta} = \beta$. For what values of β. φ, w_f, w_s, and p_0 sophisticated IO work forever as a forklift operator? For what values will she at least once work as a spoonlifter.

Theory Field Exam
January 2010

Two parties are interested in cooperating with each other. Let the payoff to party i in a given period from choosing level of cooperation (or effort) q_i in that period be $\beta(q_i, q_j) - c(q_i)$, where q_j is the level of effort chosen by the other party (*i.e.,* $j \neq i$) in that period, $\beta : \mathbb{R}_+^2 \to \mathbb{R}$, and $c : \mathbb{R}_+ \to \mathbb{R}_+$. Assume all functions are at least twice continuously differentiable in all their arguments. Assume, further, that

- Cost and marginal cost are both increasing in q; that is, $c(\cdot)$ is increasing and strictly convex. Assume $c'(0) = 0$.
- $\beta(0, 0) = 0$.
- The benefit function is symmetric; that is, for any q and q', $\beta(q, q') = \beta(q', q)$.
- $\partial\beta(q_i, q_j)/\partial q_i > 0$.
- The function β is strictly concave.
- There exists a $q_M < \infty$ such that

$$2\frac{\partial\beta(q, q)}{\partial q} - c'(q) < 0$$

  for all $q > q_M$.

(a) Prove that, for any $x > 0$, the welfare-maximizing element from the set $\{(q_1, q_2) | q_1 + q_2 = x\}$ is $(x/2, x/2)$.

Consider the following contracting technology (call it the *formal system*): Each period, the parties can write a contract that fixes their levels of cooperation (*i.e.,* q_1 and q_2). If one party breaches by choosing a level less than the agreed upon level, the law permits the victim of that breach to claim *expectation damages*—the difference between the benefit it would have received had the other party honored the contract and the benefit it actually receives. A difficulty is that the court system is imperfect: Although it can perfectly determine that a breach has occurred, it correctly identifies the breaching party with probability $\theta \in [1/2, 1)$. Whoever is identified by the court as being the breaching party must pay the other party the expectation damages. The parameter θ is common knowledge between the parties at the time they contract. Assume for questions (b) and (c) that the parties do not take into account any future interactions they may have when deciding on their contract in any given period.

(b) Prove that the first-best level of cooperation is *not* attainable under this contracting technology.

(c) Prove that welfare is increasing in θ.

Now consider a different contracting technology (call it the *informal system*). The only contracts that are feasible are relational contracts. Absent a contract, the parties choose the levels of cooperation that constitute a Nash equilibrium. Let $\delta \in (0, 1)$ be the common per-period discount factor. Assume the game is infinitely repeated.

(d) Prove that there exists a $\delta^* < 1$ such that the first-best level of cooperation is attainable every period under a relational contract provided $\delta \geq \delta^*$. Be sure to identify clearly the punishment for breach.

(e) Suppose that $\beta(q_1, q_2) = b(q_1) + b(q_2)$, where $b(\cdot)$ is such that all previously assumed properties of the function β continue to hold. Prove that for all $\delta > 0$, the parties are able to sustain a level of cooperation via relational contracts greater than the level they would play in the Nash equilibrium of a one-period game.

Now suppose that both the formal system and informal system exist (call this the *alternatives system*). In any given period, the parties decide whether they are employing the formal or informal system (they can*not* employ both).

(f) Prove there exists a $\delta^*(\theta) < 1$ such that the first-best level of cooperation is attainable every period under a relational contract provided $\delta \geq \delta^*$. Prove that $\delta^*(\cdot)$ is an increasing function. *Interpret.*

(g) Suppose that $\beta(q_1, q_2) = b(q_1) + b(q_2)$, where $b(\cdot)$ is such that all previously assumed properties of the function β continue to hold. Prove that, for all $\delta \in (0, 1)$, welfare is decreasing in θ in some neighborhood of $1/2$ (*i.e.,* for $\theta \in [1/2, t)$, $t > 1/2$). *Interpret.*

Finally, suppose that both the formal system and informal system exist. Moreover, the parties can employ both if they wish (*i.e.,* have a formal contract and exploit their ongoing relationship). Call this the *joint system.*

(h) Prove in this regime that any level of cooperation that could have been sustained in equilibrium under the alternatives system can be sustained in equilibrium in the joint system. *Interpret.*