Econometrics Field Exam

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Instructions:

- Answer all of the following questions.
- No books, notes, tables, or calculating devices are permitted.
- You have **180** minutes to answer all questions.
- Please make your answers elegant, that is, clear, concise, and, above all, correct.

[**Question 1**] [**Multiple spell duration analysis**]. Let $\{(X_{i1}, X_{i2}, Y_{i1}, Y_{i2}, A_i)\}_{i=1}^{\infty}$ be a sequence of iid random draws; A_i is unobserved. Here Y_{i1} and Y_{i2} are two durations of interest for the i^{th} random draw with X_{i1} and X_{i2} corresponding beginning-of-spell covariate vectors. Assume that the conditional hazard of exit at $Y_t = y_t$ given $\mathbf{X} = (X_1, X_2) = \mathbf{x}$ and $A = a$ equals

$$\lambda\left(y_t | \mathbf{x}, a; \theta\right) = \lambda\left(y_t; \alpha\right) \exp\left(x_t'\beta + a\right),$$

for $t = 1, 2$ and $\theta = (\alpha, \beta')'$. Here $\lambda\left(y_t; \alpha\right) = \alpha y_t^{\alpha-1}$ is the Weibull baseline hazard function with integrated hazard $\Lambda\left(y_t; \alpha\right) = \int_0^{y_t} \lambda\left(z; \alpha\right) \mathrm{d}z = y_t^{\alpha}$. You may assume that $\theta = \theta_0$, its population value, in what follows unless explicitly noted otherwise. You may assume that Y_1 and Y_2 are conditionally independent given $\mathbf{X}$ and A.

Define the bijective function $\rho\left(z_t; \theta\right) = \Lambda\left(y_t; \alpha\right) \exp\left(x_t'\beta\right)$ with $z_t = (x_t', y_t)'$. Let $\bar{\rho}\left(z, \theta\right) = \rho\left(z_1; \theta\right) + \rho\left(z_2; \theta\right)$ and $\tilde{\rho}\left(z_t, \theta\right) = \rho\left(z_t; \theta\right) / \bar{\rho}\left(Z, \theta\right)$ for $t = 1, 2$. For what follows you may use the fact that

$$\tilde{\rho}\left(Z_t, \theta\right) | \mathbf{X}, A, \bar{\rho}\left(Z, \theta\right) \sim \text{Uniform}\left[0, 1\right]$$

and

$$\bar{\rho}\left(Z, \theta\right) | \mathbf{X}, A \sim \text{Gamma}(2, e^A).$$

[a] Consider the change-of-variables $s_1 = y_1$ and $s_2 = y_2/y_1$. Show that

$$\begin{aligned} f_{S_1,S_2|X,A}\left(s_1, s_2 | x, a; \theta\right) = & \left[\exp\left(x_1'\beta + a\right)\right]^2 \alpha s_2^{\alpha-1} \exp\left(\left(x_2 - x_1\right)'\beta\right) \\ & \times \alpha s_1^{2\alpha-1} \exp\left(-s_1^{\alpha} \exp\left(x_1'\beta + a\right)\left[1 + s_2^{\alpha} \exp\left(\left(x_2 - x_1\right)'\beta\right)\right]\right). \end{aligned}$$

[b] Show that the marginal density of s_2 equals.

$$f_{S_2|X,A}\left(s_2 | x, a; \theta\right) = \frac{\alpha s_2^{\alpha-1} \exp\left(\left(x_2 - x_1\right)'\beta\right)}{\left[1 + s_2^{\alpha} \exp\left(\left(x_2 - x_1\right)'\beta\right)\right]^2}. \quad (1)$$

Does (1) contain all available information on θ? Explain.

[c] The i^{th} unit's contribution to the marginal log-likelihood based on (1) equals

$$l_i^{\mathrm{M}}(\theta) = \ln\alpha + (\alpha - 1)\ln\left(\frac{Y_{i2}}{Y_{i1}}\right) + (X_{i2} - X_{i1})'\beta - 2\ln\left[1 + \left(\frac{Y_{i2}}{Y_{i1}}\right)^{\alpha}\exp\left((X_{i2} - X_{i1})'\beta\right)\right]. \tag{2}$$

Show that (after sufficient manipulation) the score vector for β equals

$$\mathbb{S}_{\beta}^{\mathrm{M}}(Z_i) = -(X_{i2} - X_{i1})\frac{\rho(Z_{i2}, \theta) - \rho(Z_{i1}, \theta)}{\rho(Z_{i1}, \theta) + \rho(Z_{i2}, \theta)}.$$

Without directly appealing to the conditional mean zero property of the score vector show that

$$\mathbb{E}\left[\left.\mathbb{S}_{\beta}^{\mathrm{M}}(Z_i)\right|\mathbf{X}, A\right] = 0.$$

Comment on your result.

[d] Show that differentiating (2) with respect to α yields (after sufficient manipulation)

$$\mathbb{S}_{\alpha}^{\mathrm{M}}(Z_i) = \frac{1}{\alpha} - \frac{1}{\alpha}\left\{\left[\ln\rho(Z_{i2}; \theta) - \ln\rho(Z_{i1}; \theta)\right] \times \left[\frac{\rho(Z_{i2}; \theta) - \rho(Z_{i1}; \theta)}{\rho(Z_{i1}; \theta) + \rho(Z_{i2}; \theta)}\right]\right\} - \mathbb{S}_{\beta}(Z_i)'\frac{\beta}{\alpha}.$$

Without directly appealing to the conditional mean zero property of the score vector show that

$$\mathbb{E}\left[\left.\mathbb{S}_{\alpha}^{\mathrm{M}}(Z_i)\right|\mathbf{X}, A\right] = 0.$$

Comment on your result.

[e] Assume, *for the balance of this question*, that α is known to equal one such that the baseline hazard is constant. Show that

$$\mathcal{I}^{\mathrm{M}}(\beta) = \frac{1}{3}\mathbb{E}\left[\Delta X \Delta X'\right],$$

with $\Delta X = X_2 - X_1$. Is there a heterogeneity distribution for which the marginal maximum likelihood estimate is locally semi-parametrically efficient? Explain. [f] Show that

$$\frac{\partial\mathbb{E}\left[\left.Y_t\right|\mathbf{X} = \mathbf{x}, A = a\right]}{\partial x_t} = -\beta\exp\left(-x_t'\beta - a\right).$$

Further show that, recalling that $\rho(Z_t; \theta) = Y_t\exp(X_t'\beta)$,

$$\mathbb{E}\left[\left.Y_t\exp(X_t'\beta)\right|\mathbf{X} = \mathbf{x}, A = a\right] = \exp(-a).$$

[g] Interpret the estimand

$$\gamma(x_t) = \int\frac{\partial\mathbb{E}\left[\left.Y_t\right|\mathbf{X} = \mathbf{x}, A = a\right]}{\partial x_t}\pi(a)\,\mathrm{d}a,$$

and show that

$$\mathbb{E}\left[\psi\left(Z;\beta,\gamma\right)\right]=\mathbb{E}\left[-\beta\exp\left(-x_t'\beta\right)\frac{\bar{\rho}\left(Z;\theta\right)}{2}-\gamma\left(x_t\right)\right]$$

is mean zero at $\beta=\beta_0$ and $\gamma=\gamma_0$. [i] Describe a feasible estimator for γ. Is your proposal semiparametrically efficient? Why or why not? [h] An econometric genius claims that $e^A\big|\,\mathbf{X}\sim\text{Gamma}(\kappa,\lambda)$. Briefly describe an approach to estimating $\gamma\left(x_t\right)$ which exploits this information and discuss is strengths and weaknesses vis-a-vis the approach outlined above.

[**Question 2**] Suppose $\{y_t : 1\le t\le T\}$ is an observed time series generated by the model

$$y_t=\mu+u_t,\qquad u_t=\rho u_{t-1}+\varepsilon_t,\qquad t=1,\dots,T,$$

where $u_0=u_{-1}=0$ and $\varepsilon_t\sim i.i.d.\ \mathcal{N}\left(0,1\right),$ while $\mu\in\mathbb{R}$ and $\rho\in(-1,1)$ are (possibly) unknown parameters.

(a) Find the log likelihood function $\mathcal{L}\left(\mu,\rho\right)$ and, for $r\in(-1,1)$, derive $\hat{\mu}\left(r\right)=\arg\max_{\mu}\mathcal{L}\left(\mu,r\right),$ the maximum likelihood estimator of μ when ρ is assumed to equal r.

(b) Find the limiting distribution (after appropriate centering and rescaling) of the "oracle" estimator $\hat{\mu}\left(\rho\right).$

(c) Give conditions on $\hat{\rho}$ under which $\hat{\mu}\left(\hat{\rho}\right)$ asymptotically equivalent to $\hat{\mu}\left(\rho\right).$

(d) Does $\hat{\rho}=0$ satisfy the condition derived in (c)? If not, determine whether $\hat{\mu}\left(0\right)$ is asymptotically equivalent to $\hat{\mu}\left(\rho\right).$

[**Question 3**][**LASSO and Post-LASSO**] Consider an ORT model

$$Y_i=X_i\theta_0+\epsilon_i,\quad i=1,2,\dots,n,$$

where X_i are fixed (non-random) obeying an ORT condition

$$\frac{1}{n}\sum_{i=1}^{n}X_iX_i'=I_p.$$

The LASSO estimator is

$$\widehat{\theta}_L=\arg\min_{\theta\in\mathrm{R}^p}\frac{1}{n}\sum_{i=1}^{n}(Y_i-X_i'\theta)^2+\lambda\|\theta\|_1$$

(a) Give a closed-form solution to the LASSO and Post-LASSO estimators.

(b) Suppose $\theta_0=\mathbf{0}$. Characterize the minimal value of λ that correctly forces $\widehat{\theta}_L=0$.

(c) Show that $\lambda=4\sigma\sqrt{\dfrac{2\log(2p)}{n}}$ produces a LASSO estimators whose ℓ_1 norm obeys

$$\|\widehat{\theta}_L\|_1\le C\|\theta_0\|_1$$

with probability $1-(2d)^{-1}$ with some bounded C.

(d) Let $X = (D, Z)$ where D is a treatment variable and Z is a vector of controls. The object of interest is $\beta_0 = (\theta_0)_1$. Explain why the single LASSO or Post-LASSO above is not suitable for inference on β_0, and sketch an alternative approach.

[Question 4] [History of Econometrics]

Assign each of the six quotes below to one of the six listed distinguished econometricians.

1. I'd give seminars and people would say, "What's the dependent variable?" I said, "Well, a choice." But unless you can write it as $y = X\beta + \varepsilon$, people just didn't understand. I must have given pretty bad seminars.
2. We drove to Harvard in Arnold's car that consumed more oil than gasoline and met Robert Dorfman, who took us to the Harvard Faculty Club, where we dined on a horse meat steak that was the special of the day!
3. You want people who'll bring you problems, and other people who'll help you solve them.
4. Speaking from experience, I would think that a course in economics of ancient Greece would be more attractive than offering an undergraduate seminar in econometrics. Does your course satisfy some humanities requirement as well?
5. I was also interviewed by Arnold Zellner for Wisconsin. He actually gave me an offer, but I had to turn it down. Arnold still talks about that; he wishes I went to Wisconsin. Then I would have become a Bayesian.
6. ...the Bayesian bootstrap–well gee, that is the way I teach the first-year course..

Arthur Goldberger
James Powell
Chuck Manski
George Judge
Gary Chamberlain
Takeshi Amemiya