Field Exam 2014: Econometrics

August 13, 2014

Please read carefully

You have to:

- Answer **3** of the following 4 questions.

All questions and all subsections of each question have equal weight. No books, notes, tables, or calculating devices are permitted. You have **180** minutes to answer all three questions.

Please make your answers elegant, that is, clear, concise, and, above all, correct. Good luck!

Question 1

Suppose $\{y_t : 1 \leq t \leq T\}$ is an observed time series generated by the model

$$y_t = \delta t + u_t, \qquad u_t = \rho u_{t-1} + \varepsilon_t, \qquad t = 1, \ldots, T,$$

where $u_0 = 0$ and $\varepsilon_t \sim i.i.d.\ \mathcal{N}(0,1)$, while δ is a parameter of interest and $\rho \in (-1,1)$ is a (possibly) unknown nuisance parameter.

(a) Find the log likelihood function $\mathcal{L}(\delta, \rho)$ and, for $r \in (-1,1)$, derive $\hat{\delta}(r) = \arg\max_{\delta} \mathcal{L}(\delta, r)$, the maximum likelihood estimator of δ when ρ is assumed to equal r.

(b) Find the limiting distribution (after appropriate centering and rescaling) of the "oracle" estimator $\hat{\delta}(\rho)$.

(c) Give conditions on $\hat{\rho}$ under which $\hat{\delta}(\hat{\rho})$ asymptotically equivalent to $\hat{\delta}(\rho)$.

(d) Does $\hat{\rho} = 0$ satisfy the condition derived in (c)? If not, determine whether $\hat{\delta}(0)$ is asymptotically equivalent to $\hat{\delta}(\rho)$.

Question 2

Suppose that economic theory suggests that a latent dependent variable y_i satisfies a classical linear model

$$y_i^* = x_i'\beta_0 + \epsilon_i$$

but that you do not observe y_i^* over its entire range. Instead, you observe a random sample of size n of (y_i, x_i) where $y_i = \tau(y_i^*)$, where

$$\tau(y^*) = \begin{cases} \max\{0, y^*\} & if\ y^* \leq 10 \\ -1 & if\ y^* > 10 \end{cases}$$

1. Assuming that ϵ_i is normally distributed with zero mean and unknown variance σ_0^2, and is independent of x_i; derive the form of the average log-likelihood function for the unknown parameters of this problem and the form of the asymptotic distribution of the corresponding maximum likelihood estimator. Go as far as you can characterizing the asymptotic variance.

2. Suppose that the parametric form of the error distribution is unknown. Find a $\sqrt{n}$-consistent estimator of β_0, imposing a suitable stochastic restriction on the conditional distribution of ϵ_i given x_i, and without imposing a scale normalization on β_0. If possible, give an expression for the asymptotic distribution of your estimator.

3. Now suppose that y_i^* is never observed, but only the range that it falls into is observed. More specifically,

$$\tau(y^*) = \begin{cases} 0 & if\ y^* \leq 10 \\ -1 & if\ y^* > 10 \end{cases}$$

   Describe an alternative consistent estimator of β_0 under a semiparametric restriction on the conditional distribution of the errors given the regressors. Is a scale normalization on β_0 needed, or are all the components of β_0 (including the scale) identifiable under your restriction?

Question 3

Suppose $\widehat{Q}_n(\theta)$ is a measurable criterion which is twice continuously differentiable a.s. and

$$\sup_{\theta\in\Theta} |H^{-1/2}\nabla_{\theta\theta}\widehat{Q}_n(\theta)H^{-1/2} - I| = o_P(1)$$

with H is symmetric and positive definite (thus non-singular), and $\Theta \subseteq \mathbb{R}^d$.

Suppose that $\theta_0 \in \Theta$ is the true parameter, and $\hat{\theta}_n \in int(\Theta)$ is measurable and

$$\widehat{Q}_n(\hat{\theta}_n) \leq \widehat{Q}_n(\theta), \ \forall\theta \in \Theta.$$

Finally, suppose that $\hat{\theta}_n = \theta_0 + O_P(n^{-1/2})$.

1. Show that for any $\theta \in \Theta$ and $t \in \Theta$

$$\sup_{\theta} \left| \widehat{Q}_n(\theta + H^{-1/2}t) - \widehat{Q}_n(\theta) - t'H^{-1/2}\nabla_{\theta}\widehat{Q}_n(\theta) - \frac{1}{2}t't \right| = o_P(t'H^{-1}t).$$

2. Using point (1), show that for any $v \in \Theta$,

$$2\frac{\widehat{Q}_n(\hat{\theta}_n + t_n v) - \widehat{Q}_n(\hat{\theta}_n)}{t_n^2} = v'Hv + o_P(1).$$

   with $t_n \in \mathbb{R}$ and $t_n = o(1)$.

3. Suppose $\sqrt{n}\nabla_{\theta}\widehat{Q}_n(\theta_0) \Rightarrow N(0, H)$. Using point (1), show that

$$2n(\widehat{Q}_n(\theta_0) - \widehat{Q}_n(\hat{\theta}_n)) \Rightarrow \chi^2_d.$$

4. Suppose that H is *not* non-singular; but there exists a H_n such that:

$$\sqrt{n}H_n^{-1/2}\nabla_{\theta}\widehat{Q}_n(\theta_0) \Rightarrow N(0, I),$$
$$\sqrt{n}H_n^{-1/2}(\hat{\theta}_n - \theta_0) = O_P(1), \ and$$
$$\sup_{\theta} |H_n^{-1/2}\nabla_{\theta\theta}\widehat{Q}_n(\theta)H_n^{-1/2} - I| = o_P(1)$$

   Can you extend results (1)-(3) under these assumptions? Do you still obtain asymptotic normality? At root-n rate?

Question 4

Suppose $\{(y_t, x_t, z_t) : 1 \leq t \leq T\}$ is an observed time series generated by the cointegrated system

$$y_t = \beta x_t + u_t,$$

where

$$\begin{pmatrix} u_t^y \\ \Delta x_t \\ \Delta z_t \end{pmatrix} \sim i.i.d.\ \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & \sigma_{zu} \\ 0 & 1 & \sigma_{zx} \\ \sigma_{zu} & \sigma_{zx} & 1 \end{pmatrix}\right)$$

with initial conditions $x_0 = z_0 = 0$, while β is a parameter of interest and σ_{zu} and σ_{zx} are nuisance parameters. Let $\hat{\beta}_{IV} = \left(\sum_{t=1}^{T} z_t x_t\right)^{-1}\left(\sum_{t=1}^{T} z_t y_t\right)$ be the IV estimator of β that uses z_t as an instrument.

Suppose it is known that $\sigma_{zu} = 0$, but suppose σ_{zx} is unknown.

(a) Derive the maximum likelihood estimator of β and characterize its limiting distribution (after appropriate centering and rescaling).

(b) Assuming $\sigma_{zx} \neq 0$, characterize the limiting distribution (after appropriate centering and rescaling) of $\hat{\beta}_{IV}$. Is $\hat{\beta}_{IV}$ consistent if $\sigma_{zx} = 0$?

Suppose it is known that $\sigma_{zx} = 1/2$, but suppose σ_{zu} is unknown.

(c) Derive the maximum likelihood estimator of β and characterize its limiting distribution (after appropriate centering and rescaling).

(d) Assuming $\sigma_{zu} \neq 0$, characterize the limiting distribution (after appropriate centering and rescaling) of $\hat{\beta}_{IV}$. Is $\hat{\beta}_{IV}$ a consistent estimator of β?